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A comment on the K-theory meaning of the topology of gauge fixing

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Abstract: Based on the recently discovered K-theory description of D-branes in string theory a K-theory interpretation of the topology of gauge conditions for four-dimensional gauge theories, in particular 't Hooft abelian projection, is presented. The interpretation of the dyon effect in terms of K^{-1} is also discussed. In the context of type-IIA strings, D-p-branes are also interpreted as gauge fixing singularities.

Keywords: M-Theory, D-branes, Brane Dynamics in Gauge Theories.

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1. Topology of gauge fixing

In reference [1] a ghost free unitary gauge for non-abelian gauge theories was presented. The idea consist in using some extra scalar field X transforming in the adjoint representation and to fix the non-abelian part of the gauge by impossing X to be diagonal

$$X = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}. \tag{1.1}$$

The subgroup of gauge transformations under which (1.1) is invariant is just the maximal abelian subgroup, $\mathrm{U}(1)^N$ in the case of $\mathrm{U}(N)$ gauge group. The spectrum of physical degrees of freedom in this gauge are N massless photons, $\frac{1}{2}N(N-1)$ massive charged vector fields—corresponding to the off diagonal part of the A_μ gauge matrix—and the N scalar fields $\lambda_i(x)$ defined by (1.1). In addition to these degrees of freedom we have topological pointlike defects—to be identified with magnetic monopoles—corresponding to the singularities of the gauge fixing (1.1) when two eigenvalues coincide.

The topological description of these gauge fixing singularities is as follows. The condition for two eigenvalues to coincide fixes a codimension three singularity in space-time i.e. a pointlike particle if we work in four dimensions. In the vicinity of the singularity the matrix X, prior to diagonalization, can be written like [1]

$$X = \left(\begin{array}{c|cc|c} D_1 & 0 & & 0 \\ \hline 0 & \lambda+\epsilon_3 & \epsilon_1 - i\epsilon_2 & 0 \\ & \epsilon_1 + i\epsilon_2 & \lambda-\epsilon_3 & \\ \hline 0 & 0 & & D_2 \end{array}\right), \tag{1.2}$$

where D_1 and D_2 are diagonal matrices. If we concentrate our attention on the 2×2 small matrix in (1.2) we can write

$$X = \lambda I + \sum_{i=1}^{3} \sigma_i x_i \tag{1.3}$$

for σ_i the Pauli matrices. More simply $X = \sum_{i=1}^{3} \sigma_i x_i$ for $\epsilon_i(x) = x_i$ in the vicinity of the singular point defined by $\epsilon(x = 0) = 0$. Now we can easily associate the field $X = \sum_{i=1}^{3} \sigma_i x_i$ on R^3 with a magnetic monopole. Let us consider the sphere S^2 embeded in R^3. For x lying on S^2 we have

$$X^2(x) = I\,. \tag{1.4}$$

Therefore we can define two non-trivial line bundles $L_{\pm}$ on S^2 with the fiber defined by the eigenspaces in C^2

$$Xv = \pm v\,. \tag{1.5}$$

The first Chern class of these line bundles is the magnetic charge of the monopole. The physical meaning of these ghost free unitary gauges is based on the following general argument. Let us consider for simplicity an abelian Higgs model. In perturbation theory the Higgs phase is usually defined by the condition

$$< \phi >= F \tag{1.6}$$

for some non-vanishing vacuum expectation value F. The problem with (1.6) as a non-perturbative way to define the Higgs phase, is of course that ϕ is not a gauge invariant quantity. We can always choose a gauge of type $\mathrm{Re}(\phi) > 0$, $\mathrm{Im}(\phi) = 0$ in such a way that $< \phi >$ is non-vanishing regardless what condensation takes place. This ghost free gauge condition has however a problem, namely the existence of singular points where $\mathrm{Re}(\phi) = 0$. In fact around this point the gauge fixing $\mathrm{Im}(\phi) = 0$ determines a singular gauge transformation with non-vanishing vorticity. This is the topological ancestor of the well known Nielsen-Olesen vortices in abelian superconductivity. The important point here is that the existence of this topological defect is independent on the concrete Higgs condensation taking place in the system. It only depends on the topology of the gauge fixing. The difference between the Higgs and the Coulomb phase is that in the Higgs phase these vortices would be also the macroscopic degrees of freedom, while in the Coulomb phase they will condense into the vacuum. The same type of argument for the magnetic monopoles described above will differenciate the Higgs phase from the confinement phase in pure non-abelian gauge theories. The key aspect of 't Hooft definition of microscopic variables is that the topological charges associated with the topology of the gauge fixing are stable with respect to the concrete condensation taking place. So for instance in the confinement phase these magnetically charged objects exist like in the Higgs phase, the difference is that in the confinement phase they fill the vacuum creating an electric superconductor. In this note we will offer a K-theory interpretation of this gauge fixing topology. In order to gain some intuition let us before describe very briefly the K-theory interpretation of D-branes in type-II string theory from the point of view of gauge fixing topology.

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2. D-branes and K-theory

The connection of D-brane charges and RR fields with K-theory has been worked out in a series of recent works [2]–[6]. Let us start considering the description of D-p-branes (p even) in type-IIA string theory [4]. We will start with a set of N unstable filling 9-branes. The low energy spectrum on the 9-brane world volume is (up to fermionic fields) a U(N) gauge field A and an open tachyon T transforming in the adjoint representation. Based on the approach presented in the previous section it is very natural to use the tachyon field T to define a ghost free unitary gauge. If we are working in critical ten-dimensional space-time, gauge fixing singularities where two eigenvalues of T coincide, are codimension three, i.e. the natural candidates for D6-branes. Notice that we are not saying anything on the particular form of the tachyon potential. We are just considering, as we did in the previous section, a gauge where T is diagonal and we look into the singularities when two eigenvalues coincide.

The T field in the vicinity of the singularity is just given by the matrix (1.2), which in particular means that on the R^3 transversal directions we have

$$T(x) = \sum_{i=1}^{3} x_i \sigma_i \,. \tag{2.1}$$

This is in fact the definition of the tachyon field [3, 4] around the solitonic D6-brane. In order to define the charge of the D6-brane we simply repeat what we did in previous section in order to define a non-trivial line bundle on S^2, but this time replacing the field X by the tachyon field T. In this sense we notice that the D6-brane can be simply interpreted, independently of the particular form of the tachyon potential, as a topological defect of the gauge fixing in ten-dimensional gauge theory, in perfect analogy with the topological defects in the four-dimensional abelian projection. Singularities of higher codimension $2k+1$ will correspond to $2k$ equal eigenvalues. The only difference is that in this case we should use—in order to define T in the vicinity of the singularity—the Gamma matrices $\Gamma_i, i = 1, \ldots, 2k+1$. The corresponding non-trivial vector bundles on S^{2k} are defined in a similar way with the fiber given by the eigenspaces of T (see (1.5)). The dimension of these bundles is 2^{k-1}.

It was argued in [3] and developped in [4] that type-IIA D-brane charges should be interpreted in terms of the higher K-group $K^{-1}(X)$. The definition of $K^{-1}(X)$ is

$$K^{-1}(X) = \tilde{K}(SX) \,, \tag{2.2}$$

where $\tilde{K}(SX)$ is the reduced K group of SX and where SX is the reduced suspension of X.[1] The relation of type-IIA string theory and M-theory together with (2.2) strongly

[1]The reduced $\tilde{K}$ group of X is defined as follows. We first notice that the K group of a single point is just equal to Z. In fact the vector bundles on a point are simply classified by their dimension which is a natural number, the corresponding K group is therefore simply the set of integer numbers Z. If we now choose a point p in X we can map $K(X)$ into $K(p)$, the kernel of this map is the reduced $\tilde{K}$ group of X. The reduced suspension SX of a compact space X is defined as $S^1 \wedge X$, where $X \wedge Y = X \times Y / X \times y_0 \bigcup Y \times x_0$ for x_0 and y_0 base points of X and Y, respectively. The quotient space X/Y for X a compact space is defined by identification of Y to a single point. For instance $SS^n = S^{n+1}$. See [7, 8] for mathematical details.

suggest a M-theory interpretation of $K^{-1}(X)$. In type-IIB the charges of stable D-p-branes with even codimension $2k$ are related to the group $\tilde{K}(S^{2k})$. Elements in $\tilde{K}(S^{2k})$ are one to one related with the homotopy group $\pi_{2k-1}(U(2^{k-1}))$. The M-theory version of type-IIA naturally leads to describe stable D-p-branes of codimension $2k+1$ in terms of the higher group $K^{-1}(S^{2k+1})$ or equivalently $\tilde{K}(S^{2k+2})$ instead of $\tilde{K}(S^{2k+1})$, which, by Bott periodicity theorem, is zero. This means that the relevant homotopy group should be $\pi_{2k+1}(U(2^k))$ which is one to one related with $\tilde{K}(S^{2k+2})$.

The description in terms of homotopy groups is directly connected with Sen's physical picture [9] of D-branes as solitons resulting from tachyon condensation. In the case of type-IIB U(2^{k-1}) is assumed to be isomorphic with the vacuum manifold for a non-vanishing expectation value of the tachyon. In type-IIA case the equality $\pi_{2k+1}\left(U(2^k)\right) = \pi_{2k}(U(2^k)/(U(2^{k-1}) \times U(2^{k-1})))$ again relates $\tilde{K}(S^{2k+2})$ with the assumed vacuum manifold $U(2^k)/(U(2^{k-1}) \times U(2^{k-1}))$ for the corresponding tachyon.

Coming back to $K^{-1}(S^{2k+1})$ we can easily relate this group to $K^{-1}(B^{2k+1}, S^{2k})$. This follows from the representation of S^{2k+1} as the quotient space B^{2k+1}/S^{2k}.[2] In summary, we observe how the type-IIA D-brane charges with codimension $2k+1$, are in $K^{-1}(B^{2k+1}, S^{2k})$ [4]. The K-theory representation of branes of codimension $2k$ in type IIB is given by $K(B^{2k}, S^{2k-1})$.

It is important to stress that from the point of view of gauge fixing the existence of D-p-branes, with p even, in type-IIA does not depend on the particular tachyon condensation taking place. They exist as part of the physical degrees of freedom of the U(N) gauge theory associated to filling 9-branes.

3. K-theory and gauge fixing

Next we would like to describe the topology of gauge fixing using K-theory. As observed in [4] there is a different characterization of K^{-1} that does not use any extra dimension [8]. The first ingredient is that the K^{-1} group of the semigroup of trivial vector bundles on a compact space X is isomorphic to the K^{-1} group of the semigroup of locally trivial vector bundles on X. This allows us to reduce the discussion to trivial vector bundles I^n where n denotes the dimension. Elements in K^{-1} are now related to pairs (I^n, α) where α is an automorphism of I^n. More precisely we define elementary pairs (I^n, α) if the corresponding automorphism α is homotopic to the identity within the automorphisms of I^n. We say that two pairs (I^n, α) and (I^m, β) are equivalent if there exist two elementary pairs (I^p, γ_1) and (I^q, γ_2) such that $(I^n \bigoplus I^p, \alpha \bigoplus \gamma_1)$ is isomorphic to $(I^m \bigoplus I^q, \beta \bigoplus \gamma_2)$ where $\bigoplus$ represents the direct sum. In plain terms the elements in K^{-1} are going to be one to one related with the homotopy class of the automorphism α.[3]

A unitary ghost free gauge fixing of the type described in section one for a U(N) gauge theory defines in a natural way an automorphism of the corresponding vector bundle. In fact this type of gauge fixings associate in a unique way a gauge transformation to each

[2] Here we have used the relation $K(B^n, S^{n-1}) = \tilde{K}(B^n/S^{n-1})$.

[3] More precisely the isomorphism between $[X, U(\infty)]$ and $K^{-1}(X)$ maps the homotopy class in $[X, U(\infty)]$ to the automorphism α defining the corresponding element in $K^{-1}(X)$. See theorem 3.17 in [8].

point of space-time. This is specially neat in the simplest abelian Higgs model where the gauge condition $\mathrm{Im}(\phi) = 0$ fixes in each space-time point a particular U(1) rotation, namely that one killing the imaginary part of ϕ. Thus we can associate with these gauge fixings the same type of elements used in the definition of K^{-1}. In fact for a U(N) gauge theory once we fix the non-abelian part by impossing X to be diagonal, we define the automorphism

$$\alpha(x) = e^{iX(x)} . \tag{3.1}$$

The charge in K^{-1} is determined by the homotopy class of $\alpha(x)$. Therefore what this K-group classifies for gauge theories is, as expected, the topology of the gauge fixing.

In string theory the concept of K-theory as a way to describe RR charges is associated with the physical picture of stability with respect to the processes of creation and anhilation of D-branes. In the case of gauge theories we have also a concept of stability underlying the nature of the topology of gauge conditions. Namely if we pass from a U(N) to a U(M) gauge theory with $M > N$ the very meaning of abelian projection makes the charge of a magnetic monopole associated with the subgroup U(N) completly independent of M, i.e. stable under the changes of N that correspond in the gauge theory context to the type-IIA process of creation anhilation of "elementary" D-9 filling branes [4]. Moreover the topology survives in the large—N limit.

An important aspect of $K^{-1}(X)$ is to be a group i.e. to have defined the inverse. In formal terms the inverse of the element (I^n, α) is (I^n, α^{-1}).The reason for this is that there exist an homotopy [7] in $Gl(n, C)$ transforming continously the matrix

$$\begin{pmatrix} A & \\ & B \end{pmatrix} , \tag{3.2}$$

into the matrix

$$\begin{pmatrix} AB & \\ & 1 \end{pmatrix} . \tag{3.3}$$

In terms of type-IIA branes what this operation represents is the following process. Take an unstable nine filling brane with gauge bundle E carrying some lower dimensional D-brane charge. Now add another nine filling brane with gauge bundle E but carrying the opposite lower dimensional D-brane charges. The net effect is the annihilation of the lower dimensional D-brane charges. In the case of gauge theories the equivalent process is just monopole antimonopole annihilation.

4. Dyons and M-Theory

In type-IIA string theory the group K^{-1} was first interpreted from the point of view of M-theory. A natural question is of course if the topology of gauge conditions in four-dimensional gauge theories which is also related to the K^{-1} group is indicating some type of M-theory nature of four-dimensional gauge theories. The simplest answer is that the relation of the gauge fixing magnetic monopole singularities with K^{-1} is related to the well known fact that these magnetic monopoles can be electrically charged, becoming dyons. In fact, as first observed by Witten in [10], the electric charge of monopoles in the presence

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of a non-vanishing θ angle is due to the fact that in the background of the monopole the gauge transformation defined by $e^{2\pi iQ}$, where Q is the electric charge generator, have non-trivial winding number in the π_3 of the gauge group. Now we easily identify this homotopy group with the one associated with $\tilde{K}(S^4)$ that, as we have discussed in section two, can be identified with $K^{-1}(B^3, S^2)$ which is the relevant group for classifying the magnetic charge of the monopole.

A more direct way to understand the connection between $K^{-1}(X)$ and the electric charge of the dyon is in terms of the moduli of a BPS monopole [11]. As it is well known the moduli of one monopole is $M_1 = R^3 \times S^1$ where R^3 correspond to the position of the monopole and S^1 arises from the gauge transformation

$$e^{i\omega X} \tag{4.1}$$

on any solution. The generator X is the field defined by (1.1) used to fix the gauge. In the moduli space approximation [12] where we consider the effective lagrangian describing the motion on the moduli, the generator of electric charge is precisely ∂_ω for ω the S^1 angle. It is of course tempting to associate the necessity to work with K^{-1} to the structure of the monopole moduli $R^3 \times S^1$. The piece R^3 can be identified with the codimension of the singularity and the electric phase in S^1 with the "M-theory" S^1.

In gauge theories this extra S^1 is not misterious at all, simply reflects the fact that the gauge fixing (1.1) leaves unbroken the maximal abelian subgroup and that U(1) gauge transformations $e^{2\pi iX(x)}$ are nontrivial, with non-vanishing winding number on the monopole background. The previous discussion leads to conjecture that the relevant group to describe magnetic monopole singularities is $\tilde{K}(S^3 \times S^1)$, where we compactify R^3 to S^3. Namely the K-group of the compactified magnetic monopole moduli. From the M-theory point of view it seems that we promote the non-trivial gauge transformations associated with the S^1 piece of M_1 to a physical extra dimension.

5. Some remarks and speculations

The goal in reference [1] was to define a framework and a set of rules for proving quark confinement. Since we have only abelian degrees of freedom and magnetic monopoles a good starting point is to use Dirac duality for characterizing the confinement phase in terms of condensation of these magnetic monopoles. To prove that such a condensation takes place is a hard open problem that requires to know the dynamics of these monopoles and not just the topology that proves its existence. In type-IIA string theory the claim is that for instance a D6-brane is a bound state of two unstable nine filling branes. In this case it is more or less clear that the vacuum of the theory evolves from the unstable state represented by the filling nine branes into the state with stable six branes. The equivalent process in gauge theories, where we replace the six-brane by the magnetic monopoles associated with the topology of the gauge fixing, would strongly indicate permanent quark confinement i.e. condensation of these monopoles into the vacuum. What we are missing in gauge theories is the analog of the unstability of the nine filling brane in type-IIA, although the minus sign of the beta function is probably an indirect way to point out to this unstability.

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One type of D-brane we have not discussed is the 8-brane in type-IIA. This brane is difficult to characterize in terms of gauge fixing singularities. Its K-theory description would be given in terms of $K^{-1}(S^1)$ and require some $N = \infty$ limit of a discrete $Z(N)$ with N the rank of the gauge group. In four-dimensional gauge theories such a type of brane would correspond to a domain wall with "RR field strenth" $\mathrm{Tr}F \wedge F$ i.e. the topological charge.[4] It would be possible that QCD strings would appear as strings ending on these domain walls, in the same spirit that in MQCD.

If we are bold enough to follow the analogy with M-theory, we should consider the magnetic monopoles associated with the singularities of the gauge fixing as the analog of the D0-branes in type-IIA and the corresponding Matrix model (dimensional reduction to $0+1$ of four-dimensional gauge theory) as the dynamics of these magnetic monopoles. Following this way of thinking you could expect that, similarly to what happens in Matrix theory, an extra dimension (in the gauge case a five dimension), should show up dynamically in processes of monopole scattering, very much in the same way as the eleven dimension appears by comparing the scattering of D0-branes in Matrix theory with the eleven-dimensional supergravity result [13]. This gravitational description, if it exist, would be very much at the core of the holographic approach.

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